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Decomposition of the Kähler Equation

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It is shown that the Kähler equation in a four-dimensional pseudo-Riemannian manifold with Lorentzian signature decomposes uniquely into four uncoupled equations of Duffin type.

1. KLEIN–GORDON, DUFFIN, AND KÄHLER EQUATIONS

Consider a four-dimensional orientable pseudo-Riemannian manifold (M, g) with Lorentzian signature. Denote by Λ the space of smooth inhomogeneous differential forms, i.e., the direct sum of the spaces $\Lambda^{(q)}$ of homogeneous q-forms on M. Denote by $*$ the Hodge operator, by d the exterior differentiation, by $\delta \equiv *d*$ the codifferentiation, and by $\Box \equiv -(d\delta + \delta d)$ the generalized d'Alembertian.

It is known (Gel'fand *et al.*, 1961) that if ϕ is a scalar field, the Klein–Gordon equation

$$\Box\phi = m^2\phi \tag{1}$$

is equivalent to the first-order *scalar Duffin equation*, which can be written in the form

$$(d - \delta)\psi = m\psi \tag{2}$$

where ψ is an inhomogeneous differential form of type $\psi = \psi^{(0)} + \psi^{(1)}$, with $\psi^{(0)} \in \Lambda^{(0)}$ and $\psi^{(1)} \in \Lambda^{(1)}$. The relation between the solutions of the two equations is $\phi = \psi^{(0)}$.

Similarly, if ϕ is a vector field, the vector Klein–Gordon equation (1) is equivalent to the *vector Duffin equation* (or *Proca equation*) (Duffin, 1938; Kemmer, 1939), which can also be written in the form (2) provided that ψ

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is interpreted as an inhomogeneous differential form of type $\psi = \psi^{(1)} + \psi^{(2)}$, with $\psi^{(1)} \in \Lambda^{(1)}$ and $\psi^{(2)} \in \Lambda^{(2)}$. The relation between the solutions is now $\phi = \psi^{(1)}$.

For pseudovectors and pseudoscalars the Klein–Gordon equation is again equivalent to equations of the form (2) (the *pseudovector* and *pseudoscalar Duffin equations*), with the inhomogeneous form ψ of type $\psi^{(2)} + \psi^{(3)}$ in the former case and of type $\psi^{(3)} + \psi^{(4)}$ in the latter. (As above, $\psi^{(q)} \in \Lambda^{(q)}$.)

If ψ is interpreted as an inhomogeneous differential form of general type, equation (2) is the *Kähler equation* (Kähler, 1962; Talebaoui, 1995). The form given above to the four Duffin equations makes it evident that any of their solutions is also a solution of the Kähler equation (Benn and Tucker, 1987).

2. REDUCTION OF THE KÄHLER EQUATION

We shall show that, conversely, for $m \neq 0$ any solution ψ of the Kähler equation can be uniquely decomposed into a sum of four inhomogeneous forms $\psi^{(01)}$, $\psi^{(12)}$, $\psi^{(23)}$, and $\psi^{(34)}$ such that $\psi^{(q,q+1)}$ belongs to the subspace $\Lambda^{(q)} + \Lambda^{(q+1)}$ of Λ and satisfies the corresponding Duffin equation. Thus:

For $m \neq 0$ the Kähler equation is decomposable. The component equations are the Duffin equations of the four possible kinds: scalar, vector, pseudovector, and pseudoscalar.

Proof. Setting $\psi \equiv \Sigma_{q=0}^{4} \psi^{(q)}$ (where $\psi^{(q)} \in \Lambda^{(q)}$), let us write the Kähler equation more explicitly by separating its homogeneous components:

$$d\psi^{(q-1)} - \delta\psi^{(q+1)} = m\psi^{(q)} \tag{3}$$

($q = 0, 1, \ldots, 4$). For $q = 1, 2, 3$ we set

$$\psi_1^{(q)} = \psi^{(q)} + \frac{1}{m}\,\delta\psi^{(q+1)} \tag{4}$$

and

$$\psi_2^{(q)} = -\frac{1}{m}\,\delta\psi^{(q+1)} \tag{5}$$

(which is permissible since we assumed $m \neq 0$), so that

$$\psi^{(q)} = \psi_1^{(q)} + \psi_2^{(q)} \tag{6}$$

For $q = 0$ equation (3) gives $-\delta\psi^{(1)} = m\psi^{(0)}$, which can also be written

$$-\delta\psi_1^{(1)} = m\psi^{(0)} \tag{7}$$

on account of definition (4) and of the identity $\delta^2 = 0$.

For $q = 1$ equation (3) gives $d\psi^{(0)} - \delta\psi^{(2)} = m\psi^{(1)}$, which is the same as

$$d\psi^{(0)} = m\psi_1^{(1)} \tag{8}$$

by definition (5). Since one has $d\psi_1^{(1)} = (1/m)\, d^2\psi^{(0)} = 0$, equations (7) and (8) are equivalent to

$$(d - \delta)(\psi^{(0)} + \psi_1^{(1)}) = m(\psi^{(0)} + \psi_1^{(1)}) \tag{9}$$

so that the inhomogeneous form defined by

$$\psi^{(01)} \equiv \psi^{(0)} + \psi_1^{(1)}$$

is a solution of the scalar Duffin equation.

For $q = 2$ equation (3) gives $d\psi^{(1)} - \delta\psi^{(3)} = m\psi^{(2)}$, so that, by definition (4), one has $d\psi^{(1)} = m\psi_1^{(2)}$, or, on account of (8), (5), (6), and the identity $d^2 = 0$,

$$d\psi_2^{(1)} = m\psi_1^{(2)} \tag{10}$$

On the other hand, for $q = 1$ definition (5) gives $-\delta\psi^{(2)} = m\psi_2^{(1)}$, or, by $\delta^2 \equiv 0$ and definition (4),

$$-\delta\psi_1^{(2)} = m\psi_2^{(1)} \tag{11}$$

Equations (10) and (11) imply

$$(d - \delta)(\psi_2^{(1)} + \psi_1^{(2)}) = m(\psi_2^{(1)} + \psi_1^{(2)}) \tag{12}$$

since by (10) one has $d\psi_1^{(2)} = (1/m)\, d^2\psi_2^{(1)} = 0$, while (5) and the identity $\delta^2 \equiv 0$ imply $\delta\psi_2^{(1)} = 0$. Equation (12) shows that the inhomogeneous form defined by

$$\psi^{(12)} \equiv \psi_2^{(1)} + \psi_1^{(2)}$$

is a solution of the vector Duffin equation.

By iteration one gets analogous conclusions that the differential form defined by

$$\psi^{(23)} \equiv \psi_2^{(2)} + \psi_1^{(3)}$$

satisfies the Duffin pseudovector equation, and the differential form defined by

$$\psi^{(34)} \equiv \psi_2^{(3)} + \psi^{(4)}$$

satisfies the pseudoscalar Duffin equation.

It is clear from (6) and from the definitions of the forms $\psi^{(q,q+1)}$ that

$$\psi = \psi^{(01)} + \psi^{(12)} + \psi^{(23)} + \psi^{(34)} \tag{13}$$

is the asserted decomposition.

On the other hand, whenever such a decomposition of an inhomogeneous form $\psi \equiv \Sigma_{q=0}^{4} \psi^{(q)}$ into solutions of the Duffin equations is possible, the homogeneous component of degree 0 of ψ must coincide with the analogous component of $\psi^{(01)}$, and the latter determines the homogeneous component of degree 1 of $\psi^{(01)}$ (say $\psi_1^{(1)}$) via the scalar equation (1). Then the homogeneous component of degree 1 of $\psi^{(12)}$ can only be $\psi - \psi_1^{(1)}$, from which the homogeneous component of degree 2 of $\psi^{(12)}$ is determined via the vector equation (2), and so on. Thus the decomposition (13) is unique.

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